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A SIMPLIFIED METHOD OF ELASTIC-STABILITY ANALYSIS FOR THIN CYLINDRICAL SHELLS

II - MODIFIED EQUILIBRIUM EQUATION

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SUMMARY

A modified form of Donnell's equation for the equilibrium of thin cylindrical shells is derived which is equivalent to Donnell's equation but has certain advantages in physical interpretation and in ease of solution, particularly in the case of shells having clamped edges. The solution of this modified equation by means of trigonometric series and its application to a number of problems concerned with the shear buckling stresses of cylindrical shells are discussed. The question of implicit boundary conditions also is considered.

INTRODUCTION

During a general theoretical investigation of the stability of curved sheet under load, a method of analysis was developed which appears to be simpler to apply than those in general use. The development of this method is presented in two parts, of which reference 1 is the first and the present paper the second. The specific problems solved by this new method are treated in detail in other papers. (See, for example, references 2 to 7.)

In reference 1 the stability of a stressed cylindrical shell was analyzed in terms of Donnell's equation, a partial differential equation for the radial displacement w, which takes into account the effects of the axial displacement u and the circumferential displacement v. Reference 1 shows the manner in which this equation can be used to obtain relatively easy solutions to a number of problems concerning the stability of cylindrical shells with simply supported edges. The results of the solution of this equation were shown to take on a simple form by the use of the parameter k (similar to the buckling-stress coefficients for flat plates) to represent the state of stress in the shell and the parameter Z to represent the dimensions of the shell, where Z is defined by the following equations:

For a cylinder of length L

$$Z = \frac{L^2}{rt}\sqrt{1 - \mu^2}$$

and for a curved panel of width b

$$Z = \frac{b^2}{rt}\sqrt{1 - \mu^2}$$

where

r radius of curvature

t thickness of shell

and

μ Poisson's ratio for material

The accuracy of Donnell's equation was established by comparisons of the results found by its use with the results found by other methods and by experiment.

In the simplest method that has been found for solving Donnell's equation, the radial displacement w is represented by a trigonometric series expansion. This method can be used to great advantage for cylinders or curved panels with simply supported edges but leads to incorrect results when applied uncritically to cylinders or panels with clamped edges.

In the present paper an equation is derived which is equivalent to Donnell's equation but is adapted to solution for clamped as well as simply supported edges by means of trigonometric series. This modified equation retains the advantages of Donnell's equation in ease of solution and simplicity of results. The solution of the modified equation by means of the Galerkin method is explained, and the results obtained by this approach in a number of problems concerned with the shear buckling stresses of cylindrical shells are given in graphical form and discussed briefly. Boundary conditions implied by the method of solution of the modified equation are also discussed.

SYMBOLS

a	length of curved panel (longer dimension)
b	width of curved panel (shorter dimension)
a_{ij}, b_{ij}	deflection coefficients in trigonometric series
k_s	shear-stress coefficient $\left(\frac{\tau t L^2}{D\pi^2}\right.$ for cylinder; $\left.\frac{\tau t b^2}{D\pi^2}\right.$ for curved panel$\left.\right)$
k_x	direct-axial-stress coefficient $\left(\frac{\sigma_x t L^2}{D\pi^2}\right.$ for cylinder; $\frac{\sigma_x t b^2}{D\pi^2}$ for curved panel$\left.\right)$
k_y	circumferential-stress coefficient $\left(\frac{\sigma_y t L^2}{D\pi^2}\right.$ for cylinder; $\frac{\sigma_y t b^2}{D\pi^2}$ for curved panel$\left.\right)$
p	lateral pressure
r	radius of curvature of cylindrical shell
t	thickness of cylindrical shell
u	displacement in axial (x-) direction of point on shell median surface
v	displacement in circumferential (y-) direction of point on shell median surface
w	displacement in radial direction of point on shell median surface; positive outward
x	axial coordinate
y	circumferential coordinate
i, j, m, n, p, q	integers
D	plate flexural stiffness per unit length $\left(\frac{Et^3}{12(1 - \mu^2)}\right)$

E	Young's modulus of elasticity
F	Airy's stress function for median-surface stresses produced by buckle deformation $\left(\frac{\partial^2 F}{\partial y^2}\text{, stress in axial direction; } \frac{\partial^2 F}{\partial x^2}\text{, stress in circumferential direction; } -\frac{\partial^2 F}{\partial x\,\partial y}\text{, shear stress}\right)$
L	length of cylinder
Q, Q_1, Q_2	mathematical operators
R_s	shear-stress ratio; ratio of shear stress present to critical shear stress when no other stress is acting
R_x	axial-compressive-stress ratio; ratio of direct axial stress present to critical compressive stress when no other stress is acting
Z	curvature parameter $\left(\frac{L^2}{rt}\sqrt{1-\mu^2}\text{ for cylinder; } \frac{b^2}{rt}\sqrt{1-\mu^2}\text{ for curved panel or long curved strip}\right)$
λ	half wave length of buckles; measured circumferentially in cylinders and axially in long curved strips
μ	Poisson's ratio for material
σ_x	applied axial stress, positive for compression
σ_y	applied circumferential stress, positive for compression
τ	applied shear stress
τ_{cr}	critical shear stress
∇^4	operator $\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)$

∇^8 operator $\left(\nabla^4(\nabla^4) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^4\right)$

∇^{-4} inverse operator defined by equations

$$\nabla^{-4}(\nabla^4 f) = \nabla^4(\nabla^{-4} f) = f$$

THEORY

Derivation of Modified Equation

The equation of equilibrium for a flat plate may be written

$$D\nabla^4 w + t\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x\, \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) + p = 0 \tag{1}$$

where p is lateral pressure. (This equation is equivalent to equation (197) of reference 8.)

For a cylindrically curved plate having a radius of curvature r, the following pair of simultaneous equations of equilibrium may be written (as a generalization of equations (11) and (10) of reference 9):

$$D\nabla^4 w + t\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x\, \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) + p + \frac{t}{r}\frac{\partial^2 F}{\partial x^2} = 0 \tag{2}$$

$$\nabla^4 F - \frac{E}{r}\frac{\partial^2 w}{\partial x^2} = 0 \tag{3}$$

where F is Airy's stress function for the median-surface stresses produced by the buckle deformation (reference 10). Equation (2) differs from equation (1) only in the addition of the term $\frac{t}{r}\frac{\partial^2 F}{\partial x^2}$, which expresses the effect of the curvature. Equation (3) shows that, unlike flat plates, cylindrical shells experience stretching of the median surface when originally straight lines in the surface are bent slightly. Elimination of F between equations (2) and (3) by suitable differentiations and additions gives the following single equation in w for the equilibrium of cylindrical shells:

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + t\nabla^4\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) + \nabla^4 p = 0 \qquad (4)$$

Equation (4), which was first derived by Donnell (reference 11), was treated in reference 1.

An alternative method for obtaining a single equation in w for the equilibrium of a cylindrical shell is to solve equation (3) for F and substitute the result into equation (2). This procedure can readily be carried out in the following manner. Differentiation of equation (3) twice with respect to x gives

$$\nabla^4 \frac{\partial^2 F}{\partial x^2} - \frac{E}{r}\frac{\partial^4 w}{\partial x^4} = 0 \qquad (5)$$

The symbolic solution of equation (5) for $\frac{\partial^2 F}{\partial x^2}$ is

$$\frac{\partial^2 F}{\partial x^2} = \frac{E}{r}\nabla^{-4}\frac{\partial^4 w}{\partial x^4}$$

Substitution of this result into equation (2) gives

$$D\nabla^4 w + \frac{Et}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + t\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) + p = 0 \qquad (6)$$

Equation (6) is simply equation (4) modified by multiplication by the operator ∇^{-4}. In the present paper, equations (4) and (6) are referred to as Donnell's equation and the modified equation, respectively.

Advantages of Modified Equation

One of the quickest and most convenient methods for obtaining solutions of flat-plate buckling problems to any desired degree of approximation uses a Fourier series type of expansion for the deflection surface w. Both Donnell's equation and the modified equation can be solved by this method in the case of buckling problems involving curved plates having simply supported edges.

As mentioned in the "Introduction," however, Donnell's equation is not well adapted to solution by Fourier series of problems involving the stability of shells with clamped edges. The cause of the trouble appears to be that the calculation of some of the high-order derivatives found in Donnell's equation sometimes leads to divergent trigonometric series when the edges are clamped. The modified equation, however, is applicable to clamped-edge problems as well as to problems involving simply supported edges because lower-order derivatives are involved.

In addition to its advantages in the solution of problems involving shells with clamped edges, equation (6) has the additional advantage that each term has a definite physical significance: The first term gives the restoring force per unit area of the deflected surface due to bending and twisting stiffnesses; the second term gives the restoring force per unit area due to stretching stiffness; and the remaining terms give the deflecting forces per unit area due to applied loads. Because of these advantages, the modified equation was adopted for general use in references 2 to 7.

Both Donnell's equation and the modified equation result in the same critical stresses for simply supported cylindrical shells, and the two methods require essentially equivalent mathematical processes. (See appendix.) The characteristics of solutions by means of Donnell's equation in the case of simply supported shells (reference 1) - namely, the theoretical cylinder parameters, the simplicity of calculations and results, and the implied boundary conditions on u and v - are characteristics, also, of solutions by means of the modified equation. The same characteristics, except for a change in the implied boundary conditions on u and v, also apply to solutions of clamped-edge shell problems by means of the modified equation. This change is discussed in the section entitled "Boundary Conditions."

Solution of Modified Equation by Galerkin Method

An approximate method of solving vibration and buckling problems closely paralleling that of Ritz was introduced in 1915 by Galerkin. (See, for example, references 12 and 13.) The main distinction between the Ritz and Galerkin methods is that the Ritz method begins with an energy expression; whereas the Galerkin method begins with an equation of equilibrium. The Galerkin method is readily adaptable to the solution of equation (6) and is now described briefly.

Let the equation of equilibrium be written

$$Q(w) + p = 0 \tag{7}$$

where p is lateral pressure and Q is some operator in x and y which for the purposes of this paper is taken to be linear. According to the Galerkin method, the equation may be solved by expanding the unknown function w in terms of a suitable set of functions $f_i(x)g_j(y)$, each of which satisfies the boundary conditions but not in general the equation of equilibrium:

$$w = \sum_i \sum_j a_{ij} f_i(x) g_j(y) \tag{8}$$

Substitution of this expression for w into equation (7) gives the following equation:

$$\sum_i \sum_j a_{ij} Q\left[f_i(x) g_j(y)\right] + p = 0 \tag{9}$$

Because the functions $f_i(x)g_j(y)$ were chosen to satisfy the boundary conditions rather than the equation of equilibrium, equation (9) cannot, in general, be satisfied identically by any choice of the coefficients a_{ij}. These coefficients can be chosen, however, to assure the vanishing of certain weighted averages of the left-hand side of equation (9). The weighting functions used in the Galerkin method are the original expansion functions, so that the following simultaneous equations for determining the coefficients a_{ij} are obtained:

$$\sum_i \sum_j B_{mnij} a_{ij} = 0 \tag{10}$$

$$(m=1,2,3, \ldots; \; n=1,2,3, \ldots)$$

where

$$B_{mnij} = \int\int f_m(x)g_n(y)\left\{Q\left[f_i(x)g_j(y)\right] + p\right\}dx\,dy \tag{11}$$

The simultaneous set of linear algebraic equations in the unknown coefficients a_{ij} (equation (10)), obtained by using the original expansion functions as weighting functions, is ordinarily the same set which would be found by the Ritz method, if the same series expansion for w were used. A solution of any desired degree of accuracy may therefore be obtained by the Galerkin method.

In applying the Galerkin method to equation (6) by use of Fourier series expansion for w, expressions of the type

$$\nabla^{-4}\sum_i\sum_j a_{ij}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

must be evaluated. The operator ∇^{-4}, the inverse of ∇^4, simply introduces into the denominator of each term of the series the expression that comes into the numerator if ∇^4 is applied. Thus,

$$\nabla^{-4}\sum_i\sum_j a_{ij}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} =$$

$$\sum_i\sum_j \frac{a_{ij}}{\left[\left(\frac{\pi i}{a}\right)^2 + \left(\frac{\pi j}{b}\right)^2\right]^2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b} \tag{12}$$

This result may readily be verified by applying the operator ∇^4 to each side of equation (12).

In writing equation (12) the quantity $\nabla^{-4}f$, as defined by the equation

$$\nabla^4\nabla^{-4}f = f$$

was tacitly assumed to be unique. The quantity actually is not unique; any number of terms which vanish when operated upon by ∇^4 could be added to the right-hand side of equation (12). The omission of such terms makes the present analysis parallel to the analysis using Donnell's equation (see reference 1) and implies certain boundary conditions on u and v, which are discussed in a subsequent section entitled "Boundary Conditions."

Deflection Functions

Simply supported edges.- For simply supported cylindrical shells, the following series expansions for w may be used to represent the buckle deformation to any desired degree of accuracy (in these functions, x is the coordinate in the axial direction and y, the coordinate in the circumferential direction):

(1) Rectangular curved plate (axial dimension a and circumferential dimension b)

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{13}$$

(2) Curved strip long in the axial direction (circumferential width b and axial wave length 2λ)

(a) Direct stresses only

$$w = \sin \frac{\pi x}{\lambda} \sum_{m=1}^{\infty} a_m \sin \frac{m\pi y}{b} \tag{14}$$

(b) Shear stress with or without addition of direct stress

$$w = \sin \frac{\pi x}{\lambda} \sum_{m=1}^{\infty} a_m \sin \frac{m\pi y}{b} + \cos \frac{\pi x}{\lambda} \sum_{m=1}^{\infty} b_m \sin \frac{m\pi y}{b} \tag{15}$$

(3) Complete cylinder (length L and circumferential wave length 2λ)

(a) Direct stresses only

$$w = \sin \frac{\pi y}{\lambda} \sum_{m=1}^{\infty} a_m \sin \frac{m\pi x}{L} \tag{16}$$

(b) Shear stress with or without addition of direct stress

$$w = \sin \frac{\pi y}{\lambda} \sum_{m=1}^{\infty} a_m \sin \frac{m\pi x}{L} + \cos \frac{\pi y}{\lambda} \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{L} \tag{17}$$

Clamped edges.- Probably the simplest method of treating cylindrical shells with clamped edges is to employ the expansions in equations (13) to (17) modified by substituting functions of the type

$$\varphi_m(x) = \sin \frac{m\pi x}{a} \sin \frac{\pi x}{a} = \frac{1}{2}\left[\cos \frac{(m-1)\pi x}{a} - \cos \frac{(m+1)\pi x}{a}\right] \tag{18}$$

wherever functions of the type $\sin \frac{m\pi x}{a}$ appear, with a similar substitution for functions of y (all terms involving summation subscripts m and n are thus changed; terms involving λ, such as $\sin \frac{\pi x}{\lambda}$, remain unchanged). The functions $\varphi_m(x)$ form a complete set so that finite expansions for w of the type suggested for shells with clamped edges as well as those for shells with simply supported edges may be used to represent the buckle deformation to any desired degree of accuracy.

Boundary Conditions

Simply supported edges.- Appendix D of reference 1 shows that, if the buckling stress of a simply supported shell is found by means of the expansions for w given in the preceding section entitled "Deflection Function," the boundary or edge conditions implied for the median-surface displacements u and v are zero displacement along each of the edges of a cylinder or curved panel and free displacement normal to each edge. (Although the proof given used equation (4), the proof could equally well have been based on equation (6).)

The boundary conditions for simple support may thus be written, at a curved edge (x = constant),

$$w = \frac{\partial^2 w}{\partial x^2} = v = \frac{\partial^2 F}{\partial y^2} = 0 \tag{19}$$

and, at a straight edge (y = constant),

$$w = \frac{\partial^2 w}{\partial y^2} = u = \frac{\partial^2 F}{\partial x^2} = 0 \tag{20}$$

Clamped edges.- By a method similar to that in appendix D of reference 1, solutions using the functions suggested in the preceding section for the treatment of clamped edges can be shown to correspond to the boundary conditions - zero displacement normal to an edge and free displacement along an edge.

The boundary conditions for clamped edges therefore become, at a curved edge (x = constant),

$$w = \frac{\partial w}{\partial x} = u = \frac{\partial^2 F}{\partial x^2} = 0 \tag{21}$$

and, at a straight edge (y = constant),

$$w = \frac{\partial w}{\partial y} = v = \frac{\partial^2 F}{\partial y^2} = 0 \tag{22}$$

Discussion.- As mentioned in reference 1, the boundary conditions implied for u and v in the case of simply supported edges are appropriate for cylinders or panels bounded by light bulkheads or deep stiffeners, which are stiff in their own planes but may be readily warped out of their planes.

The boundary conditions on u and v appropriate for a clamped edge would seem to be zero displacement normal to the edge and zero, rather than free, displacement along the edge. Comparison of critical stresses for shells with clamped edges found by the method in the present paper with critical stresses found by the method in references 9 and 14, giving boundary conditions $u = v = 0$, however, indicates that the imposition of the added requirement of zero displacement along the edge ordinarily has very little effect on the critical stresses.

A less satisfactory method for solving problems concerning shells with clamped edges involves the use of functions of the type

$$\frac{1}{m} \sin \frac{m\pi x}{a} - \frac{1}{m + 2} \sin \frac{(m + 2)\pi x}{a}$$

instead of those described by equation (18). In this method, the functions used are those for simple support taken in such combinations that the edge slope is zero. Use of such functions leads to the same boundary conditions on u and v as were described for simply supported edges; at the edge y = constant, for example, the boundary conditions become

$$w = \frac{\partial w}{\partial y} = u = \frac{\partial^2 F}{\partial x^2} = 0 \tag{23}$$

The use of these functions to represent shells with clamped edges is not recommended, however, for the following reasons: The associated boundary conditions seem to be artificial and unlikely to be reproduced even approximately in actual construction; the method leads in some cases to solutions that differ considerably from the solution for ideal clamped-edge conditions in which $u = v = 0$; and the solutions obtained generally converge rather poorly.

APPLICATIONS AND DISCUSSION

Among the more difficult shell-stability problems to treat theoretically are those which involve shear stresses. In fact, until 1934, when Donnell's paper on critical shear stress of a cylinder in torsion was published (reference 11), such problems were generally regarded as impracticable to solve. In order to illustrate the type of solution to be found by the method of analysis just outlined and the effect of boundary conditions on critical stresses, the results obtained for a number of shell stability problems involving shear stresses are reproduced and discussed briefly here. The problems treated are summarized in table I.

Critical shear stress of long curved strip.- The critical shear stress for a long plate with transverse curvature is given by the equation

$$\tau_{cr} = k_s \frac{\pi^2 D}{b^2 t}$$

where k_s is a dimensionless coefficient, the value of which depends upon the dimensions of the strip, Poisson's ratio for the material, and the type of edge support. In figure 1 (fig. 1 of reference 2) the shear-stress coefficient k_s is given for plates with simply supported edges and with clamped edges. This solution for simply supported edges coincides with that given by Kromm (reference 15).

As indicated in the previous section entitled "Boundary Conditions," the solution corresponding to the boundary conditions of equation (23) (dashed curve of fig. 1) is poorly convergent and deviates appreciably from the results for completely fixed edges. Figure 1 shows this poor convergence in the limiting case of a flat plate, for which the critical stress is independent of boundary conditions on u and v. Even a tenth-order determinant led to a result that is 7 percent above the true solution; whereas the result using a fourth-order determinant obtained with the deflection functions recommended for clamped edges is only 1 percent above.

In figure 2 (fig. 2 of reference 2) the solutions given in figure 1 are compared with the results given by Leggett (reference 9) for simply supported and clamped edges with $u = v = 0$ at each edge. Throughout the range for which they are given, Leggett's results for clamped edges differ only slightly from those of the present paper. On the other hand, the previously mentioned discrepancy between the results for completely fixed edges ($u = v = 0$) and those for the boundary conditions of equation (23) (dashed curve) may be inferred from this figure to be considerable for large values of Z. A minimum measure of this discrepancy is the distance between the clamped-edge curves for $v = 0$ and for $u = 0$ in figure 2, since Leggett's curve must always lie above the curve for $v = 0$.

The reason for the marked increase in buckling stress of simply supported curved strips when the edges are restrained against circumferential displacement during buckling is discussed in reference 2.

Critical shear stress of cylinder in torsion.- The critical shear stress of a cylinder subjected to torsion is given by the equation

$$\tau_{cr} = k_s \frac{\pi^2 D}{L^2 t}$$

In figure 3 (fig. 1 of reference 3) the values of k_s are given for cylinders with simply supported edges (boundary conditions of equation (19)) and cylinders with clamped edges (boundary conditions of equation (21)). At high values of Z, the values of k_s for thick cylinders are given by special curves for various values of $\frac{r}{t}\sqrt{1 - \mu^2}$, as discussed in reference 1. At values of Z greater than about 100 only a small increase in buckling stress is caused by clamping the edges. The results indicated in figure 3 are in very close agreement with Donnell's results for the same problem, except in the range $5 < Z < 500$ where the somewhat lower curves of the present paper represent a more accurate solution.

Reference 1 shows that boundary conditions imposed upon u and v at the curved edges of a panel or cylinder have an almost insignificant effect on the buckling stresses, whereas conditions imposed on v at the straight edges may be quite important. Comparison of figure 1, in which boundary conditions on straight edges are considered, with figure 3, in which conditions on curved edges are considered, indicates that a similar situation exists with respect to restraint against edge rotation.

Critical shear stress of curved panel.- The values of k_s giving the critical shear stresses of simply supported curved rectangular panels are given in figures 4 and 5 (figs. 1 and 2, respectively, of reference 4). The corresponding boundary conditions on u and v are zero displacement parallel to the edges and free warping normal to the edges. Figure 4 indicates that as the curvature parameter Z increases, the critical shear stresses of panels having a circumferential dimension greater than the axial dimension approach those for a cylinder. Figure 5 indicates that, as the curvature parameter Z increases, the critical shear stresses for panels having an axial dimension greater than the circumferential dimension deviate more and more from the critical shear stress for an infinitely long curved plate. Reference 4 shows that the reason for this deviation in figure 5 is that at high curvatures the buckling stresses of these panels, as well as those of figure 4, approach those of the cylinder obtained by extending the circumferential dimensions of the panels.

The effects of boundary conditions in the limiting cases of infinitely long curved strips (fig. 1) and of complete cylinders (fig. 3) suggest that the curves of figure 4 are substantially independent of edge restraint at large values of Z but that the curves of figure 5 would be considerably affected by a change in edge restraint.

Long curved strips under combined shear and direct axial stress. Reference 5 shows that the theoretical interaction curve for a long curved strip under combined shear stress and direct axial stress is approximately parabolic when the edges are either simply supported or clamped, regardless of the value of Z. This parabola is given by the formula

$$R_s^2 + R_x = 1$$

where R_s and R_x are the shear-stress and compressive-stress ratios, respectively.

At high values of Z curved strips, like cylinders, buckle at compressive stresses considerably below the theoretical critical stresses. In order to take this condition into account, certain modifications in the theoretical results are proposed in reference 5 for use in design.

Cylinders under combined shear and direct axial stress.- The theoretically determined combinations of shear stress and direct axial stress which cause a cylinder with simply supported and clamped edges to buckle are shown in figure 6 (fig. 1 of reference 6). Considerable variation in the shape of the interaction curves occurs for low values of Z. For high values of Z the interaction curves for either simply supported or clamped edges are similar to the curve for $Z = 30$.

Because cylinders actually buckle at a small fraction of their theoretical critical compressive stress, the theoretical interaction curves of figure 6 cannot be expected to be in satisfactory agreement with experiment when a very appreciable amount of compression is present. For semiempirical curves and a check of available test data, see reference 6.

CONCLUDING REMARKS

A previous investigation showed how Fourier series type solutions of Donnell's equation can be used to simplify greatly the stability analysis of thin cylindrical shells with simply supported edges. The present paper shows that the restriction to simply supported edges can be removed by the introduction of a new equation which is equivalent to Donnell's equation but is better adapted to solution by Fourier series. This modified equation can be solved for the buckling stresses of curved sheet having either simply supported or clamped edges by established methods essentially equivalent to those in use for flat sheet. This approach permits a simple and straightforward solution to be given for a number of problems previously considered rather formidable.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
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APPENDIX

COMPARISON OF RESULTS OBTAINED BY USING DONNELL'S EQUATION AND THE MODIFIED EQUATION IN THE STABILITY ANALYSIS OF SIMPLY SUPPORTED CURVED PANELS

Solution of Donnell's Equation

Donnell's equation expressing the equilibrium of a curved panel under median-surface stresses can be written in general form as

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + \sigma_x t\nabla^4 \frac{\partial^2 w}{\partial x^2} + 2\tau t\nabla^4 \frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y t\nabla^4 \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{(A1)}$$

where x is the axial coordinate and y the circumferential coordinate. Division of equation (A1) by D and the introduction of the dimensionless stress coefficients k_x, k_y, and k_s, and the curvature parameter Z results in the following equation:

$$\nabla^8 w + \frac{12Z^2}{b^4}\frac{\partial^4 w}{\partial x^4} + k_x\frac{\pi^2}{b^2}\nabla^4 \frac{\partial^2 w}{\partial x^2} + 2k_s\frac{\pi^2}{b^2}\nabla^4 \frac{\partial^2 w}{\partial x\,\partial y} + k_y\frac{\pi^2}{b^2}\nabla^4 \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{(A2)}$$

where

$$k_x = \sigma_x\frac{b^2 t}{\pi^2 D}$$

$$k_s = \tau\frac{b^2 t}{\pi^2 D}$$

$$k_y = \sigma_y\frac{b^2 t}{\pi^2 D}$$

and

$$Z = \frac{b^2}{rt}\sqrt{1 - \mu^2}$$

Equation (A2) can be represented by

$$Q_1(w) = 0 \tag{A3}$$

where Q_1 is defined as the operator

$$\nabla^8 + \frac{12Z^2}{b^4}\frac{\partial^4}{\partial x^4} + k_x\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2}{\partial x^2} + 2k_s\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2}{\partial x\,\partial y} + k_y\frac{\pi^2}{b^2}\nabla^4\frac{\partial^2}{\partial y^2}$$

The equation of equilibrium (equation (A3)) is solved by using the Galerkin method as described in the section entitled "Theory." In applying this method the unknown deflection w is represented in terms of a set of functions (see equation (8)), each of which satisfies the boundary conditions but not in general the equation of equilibrium. A suitable set of functions of this type, which satisfies the boundary conditions for simple support, is

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A4}$$

where the origin is taken at a corner of the plate. Substituting in equations (10) and (11)

$$f_m(x) = \sin\frac{m\pi x}{a}$$

$$g_n(y) = \sin\frac{n\pi y}{b}$$

and

$$Q = Q_1$$

and performing the integration over the whole plate (limits $x = 0, a$; $y = 0, b$) gives the set of equations

$$a_{mn}\left[\left(m^2 + n^2\frac{a^2}{b^2}\right)^4 + \frac{12Z^2m^4a^4}{\pi^4b^4}\right.$$

$$\left. - k_xm^2\left(m^2 + n^2\frac{a^2}{b^2}\right)^2 - k_yn^2\frac{a^2}{b^2}\left(m^2 + n^2\frac{a^2}{b^2}\right)^2\right]$$

$$+ \frac{32k_s}{\pi^2}\frac{a^3}{b^3}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} a_{pq}\frac{\left(p^2 + q^2\frac{a^2}{b^2}\right)^2 mnpq}{(m^2 - p^2)(n^2 - q^2)} = 0 \qquad \text{(A5)}$$

where $m=1,2,3, \ldots$, $n=1,2,3, \ldots$, and p and q take only those values for which $m \pm p$ and $n \pm p$ are odd numbers.

Equation (A5) represents an infinite set of homogeneous linear equations involving the unknown deflection coefficients a_{ij}. In order for the deflection coefficients to have values other than zero, that is, in order for the panel to buckle, the determinant of the coefficients of the unknown deflection coefficients a_{ij} must vanish. This determinant can be factored into two subdeterminants, one involving the unknown deflection coefficients a_{ij} for which $i \pm j$ is odd and the other involving those coefficients for which $i \pm j$ is even. Buckling occurs, therefore, when either of the two subdeterminants vanishes. Only the buckling criterion involving the even subdeterminant is treated here. This criterion is

$$
\begin{array}{c|cccccc|l}
 & a_{11} & a_{13} & a_{22} & a_{31} & a_{33} & \cdots & \\
m=1,\ n=1 & M_{11} & 0 & +\frac{4}{9}\left(4+4\frac{a^2}{b^2}\right)^2 & 0 & 0 & \cdots & \\
m=1,\ n=3 & 0 & M_{13} & -\frac{4}{5}\left(4+4\frac{a^2}{b^2}\right)^2 & 0 & 0 & \cdots & \\
m=2,\ n=2 & +\frac{4}{9}\left(1+\frac{a^2}{b^2}\right)^2 & -\frac{4}{5}\left(1+9\frac{a^2}{b^2}\right)^2 & M_{22} & -\frac{4}{5}\left(9+\frac{a^2}{b^2}\right)^2 & +\frac{36}{25}\left(9+9\frac{a^2}{b^2}\right)^2 & \cdots & \\
m=3,\ n=1 & 0 & 0 & -\frac{4}{5}\left(4+4\frac{a^2}{b^2}\right)^2 & M_{31} & 0 & \cdots & = 0 \quad \text{(A6)} \\
m=3,\ n=3 & 0 & 0 & +\frac{36}{25}\left(4+4\frac{a^2}{b^2}\right)^2 & 0 & M_{33} & \cdots & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & &
\end{array}
$$

where

$$
M_{mn} = \frac{\pi^2 b^3}{32 k_s a^3}\left[\left(m^2 + n^2\frac{a^2}{b^2}\right)^4 + \frac{12Z^2 m^4 a^4}{\pi^4 b^4} - k_x m^2\left(m^2 + n^2\frac{a^2}{b^2}\right)^2 - k_y n^2\frac{a^2}{b^2}\left(m^2 + n^2\frac{a^2}{b^2}\right)^2\right]
$$

Division of each column of the determinant in equation (A6) by the proper

$$
\left(i^2 + j^2\frac{a^2}{b^2}\right)^2
$$

gives the simplified equation

$$\begin{array}{cccccc} a_{11}\left(1+\frac{a^2}{b^2}\right)^2 & a_{13}\left(1+9\frac{a^2}{b^2}\right)^2 & a_{22}\left(4+4\frac{a^2}{b^2}\right)^2 & a_{31}\left(9+\frac{a^2}{b^2}\right)^2 & a_{33}\left(9+9\frac{a^2}{b^2}\right)^2 & \cdots \end{array}$$

$$\begin{vmatrix} N_{11} & 0 & +\frac{4}{9} & 0 & 0 & \cdots \\ 0 & N_{13} & -\frac{4}{5} & 0 & 0 & \cdots \\ +\frac{4}{9} & -\frac{4}{5} & N_{22} & -\frac{4}{5} & +\frac{36}{25} & \cdots \\ 0 & 0 & -\frac{4}{5} & N_{31} & 0 & \cdots \\ 0 & 0 & +\frac{36}{25} & 0 & N_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{vmatrix} = 0 \qquad \text{(A7)}$$

where

$$N_{mn} = \frac{\pi^2 b^3}{32k_s a^3}\left[\left(m^2 + n^2\frac{a^2}{b^2}\right)^2 + \frac{12Z^2m^4a^4}{\pi^4b^4\left(m^2 + n^2\frac{a^2}{b^2}\right)^2} - k_x m^2 - k_y n^2\frac{a^2}{b^2}\right]$$

The vanishing of this determinant is the criterion for the symmetrical buckling of the shell. The same buckling criterion results from the use of the modified equation, as is shown in the following section.

Solution of Modified Equation

The modified equation expressing the equilibrium of a curved panel under median-surface stresses in general form is

$$D\nabla^4 w + \frac{Et}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + \sigma_x t\frac{\partial^2 w}{\partial x^2} + 2\tau t\frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y t\frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{(A8)}$$

Division of equation (A8) by D and simplification of the result gives the following equation:

$$\nabla^4 w + \frac{12Z^2}{b^4}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + k_x\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial x^2} + 2k_s\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial x\,\partial y} + k_y\frac{\pi^2}{b^2}\frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{(A9)}$$

Equation (A9) can be represented by

$$Q_2(w) = 0 \qquad \text{(A10)}$$

where Q_2 is defined as the operator

$$\nabla^4 + \frac{12Z^2}{b^4}\nabla^{-4}\frac{\partial^4}{\partial x^4} + k_x\frac{\pi^2}{b^2}\frac{\partial^2}{\partial x^2} + 2k_s\frac{\pi^2}{b^2}\frac{\partial^2}{\partial x\,\partial y} + k_y\frac{\pi^2}{b^2}\frac{\partial^2}{\partial y^2}$$

By use of the Galerkin method and by use of the expression for w given in equation (A4), the following set of equations analogous to equations (A5) are obtained

$$a_{mn}\left[\left(m^2 + n^2\frac{a^2}{b^2}\right)^2 + \frac{12Z^2m^4a^4}{\pi^4b^4\left(m^2 + n^2\frac{a^2}{b^2}\right)^2} - k_xm^2 - k_yn^2\frac{a^2}{b^2}\right] + \frac{32k_sa^3}{\pi^2b^3}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}a_{pq}\frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} = 0 \qquad \text{(A11)}$$

where $m=1,2,3, \ldots$, $n=1,2,3, \ldots$, and p and q take only those values such that $m \pm p$ and $n \pm q$ are odd numbers.

As in the case of the solution of Donnell's equation, the stability determinant representing equations (A11) can be factored into an even and an odd subdeterminant. The even one is

$$
\begin{array}{c}
 \\ m=1,\ n=1 \\ m=1,\ n=3 \\ m=2,\ n=2 \\ m=3,\ n=1 \\ m=3,\ n=3 \\ \cdot \\ \cdot \\ \cdot
\end{array}
\begin{array}{c}
\begin{array}{cccccc} a_{11} & a_{13} & a_{22} & a_{31} & a_{33} & \cdots \end{array} \\
\left|\begin{array}{cccccc}
N_{11} & 0 & +\frac{4}{9} & 0 & 0 & \cdots \\
0 & N_{13} & -\frac{4}{5} & 0 & 0 & \cdots \\
+\frac{4}{9} & -\frac{4}{5} & N_{12} & -\frac{4}{5} & +\frac{36}{25} & \cdots \\
0 & 0 & -\frac{4}{5} & N_{31} & 0 & \cdots \\
0 & 0 & +\frac{36}{25} & 0 & N_{33} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot &
\end{array}\right|
\end{array}
= 0 \qquad \text{(A12)}
$$

The stability determinant (equation (A12)) obtained from the modified equation is identical with the simplified stability determinant (equation (A7)) obtained by use of Donnell's equation. This identity holds for the odd as well as the even determinants.

Although the stability determinants obtained by use of the two equations are identical and yield identical buckling loads, the determinant in equation (A7) consists of the coefficients of $a_{ij}\left(i^2 + j^2 \frac{a^2}{b^2}\right)^2$, whereas the determinant in equation (A12) consists of the coefficients of a_{ij}. Accordingly, although the buckling loads found by the two methods are the same, the buckle patterns are different. Of the two buckle patterns the one found by the use of the modified equation is believed to be correct. This conclusion has been verified for the limiting case of a flat plate (Z=0).

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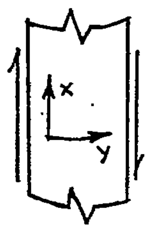
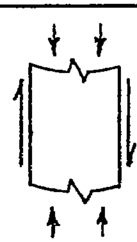
TABLE I.- INDEX OF PROBLEMS TREATED

Problem	Figure	Reference	Edge Condition
	1	2	Simply supported ($u=0$, $v\neq0$) Clamped ($u\neq0$, $v=0$) Clamped ($u=0$, $v\neq0$)
	2	2	Simply supported ($u=0$, $v\neq0$) Clamped ($u\neq0$, $v=0$) Clamped ($u=0$, $v\neq0$)
		9 (Leggett)	Simply supported ($u=v=0$) Clamped ($u=v=0$)
	3	3	Simply supported Clamped
	4	4	Simply supported
	5	4	Simply supported
	Not shown	5	Simply supported Clamped
	6(a)	6	Simply supported
	6(b)	6	Clamped

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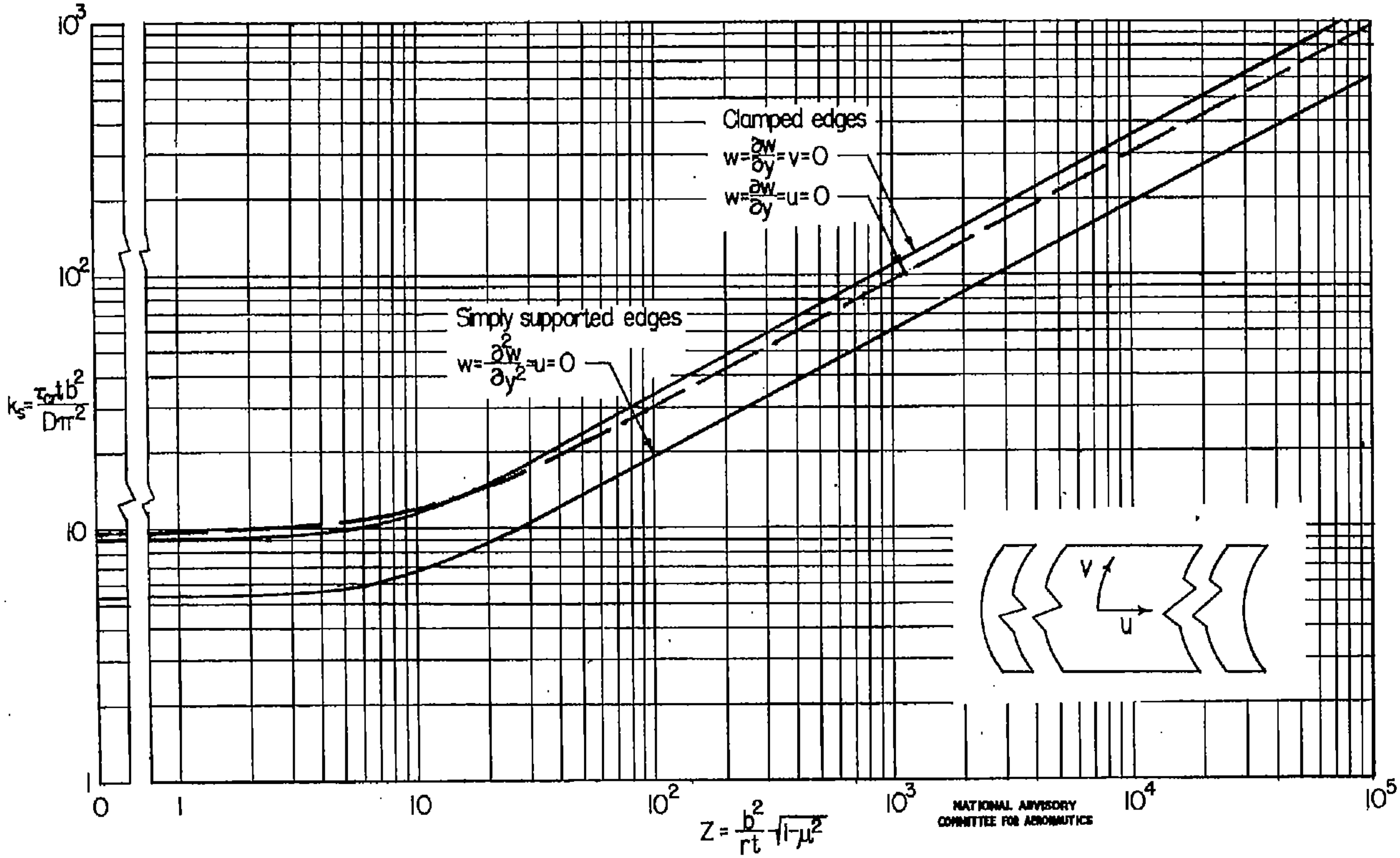


Figure 1.- Critical-shear-stress coefficients for a long curved strip. (Fig. 1 of reference 2.)

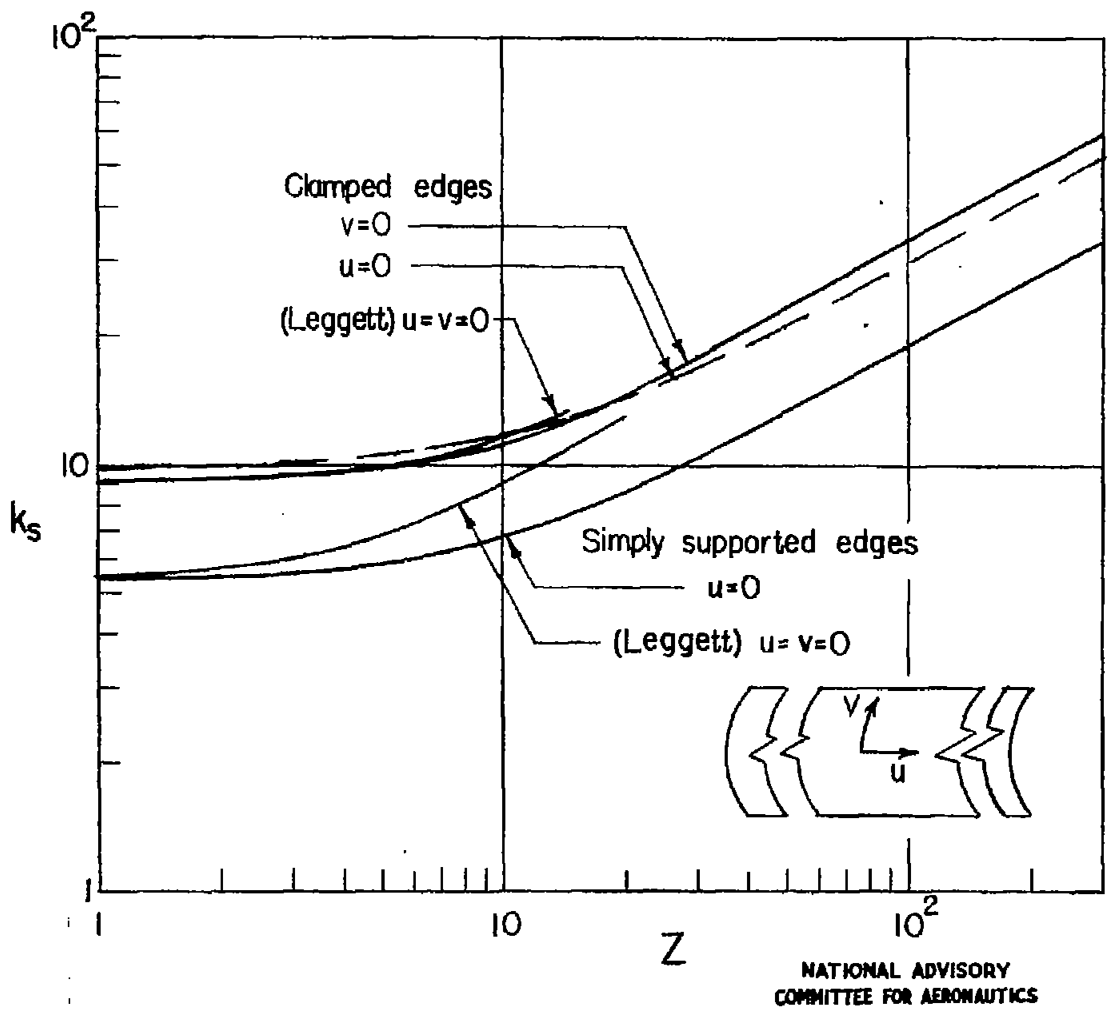


Figure 2.- Comparison of Leggett's solutions with present solutions for critical-shear-stress coefficients of a long curved strip. (Fig. 2 of reference 2.)

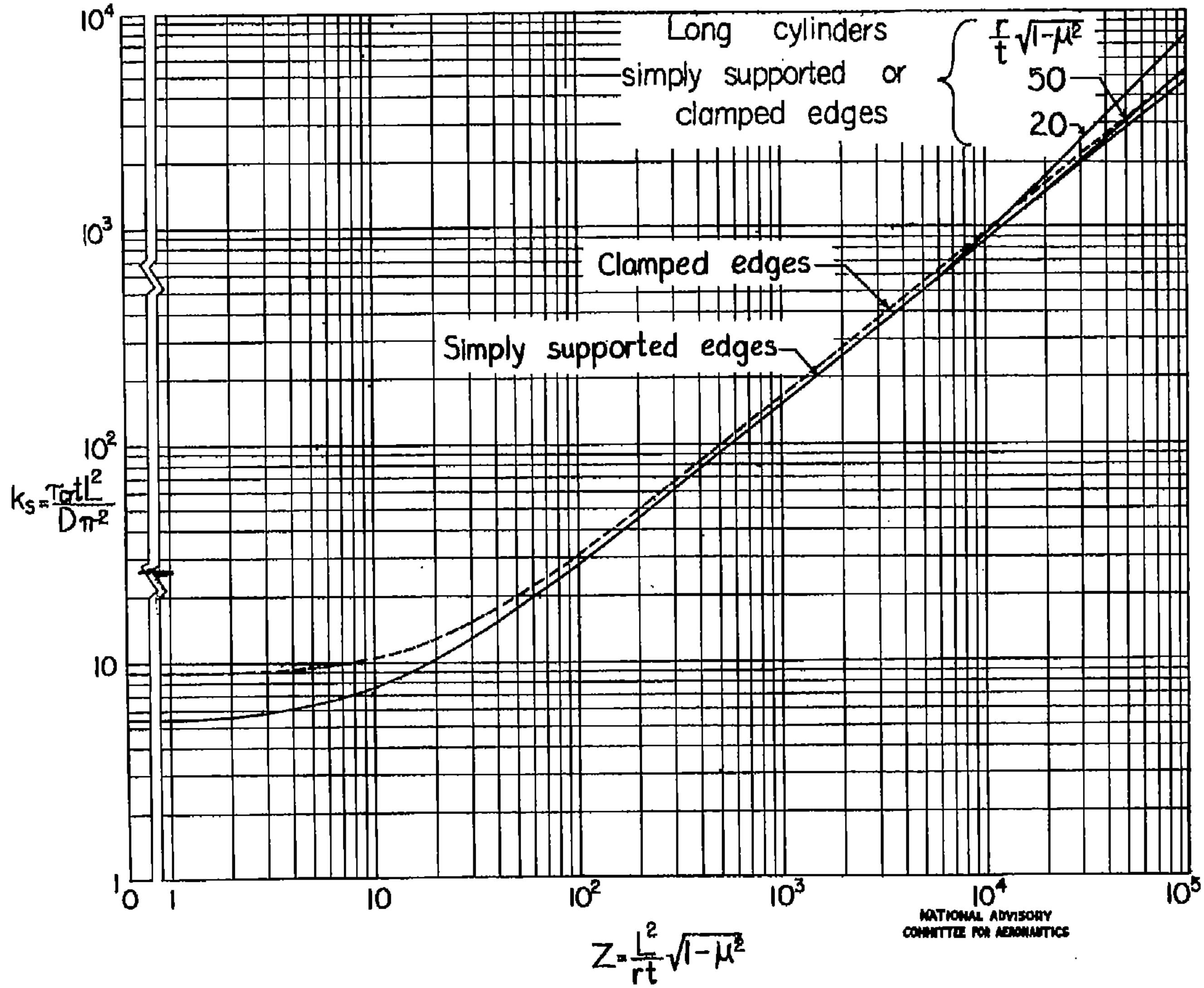


Figure 3.- Critical-shear-stress coefficients for cylinders in torsion. (Fig. 1 of reference 3.)

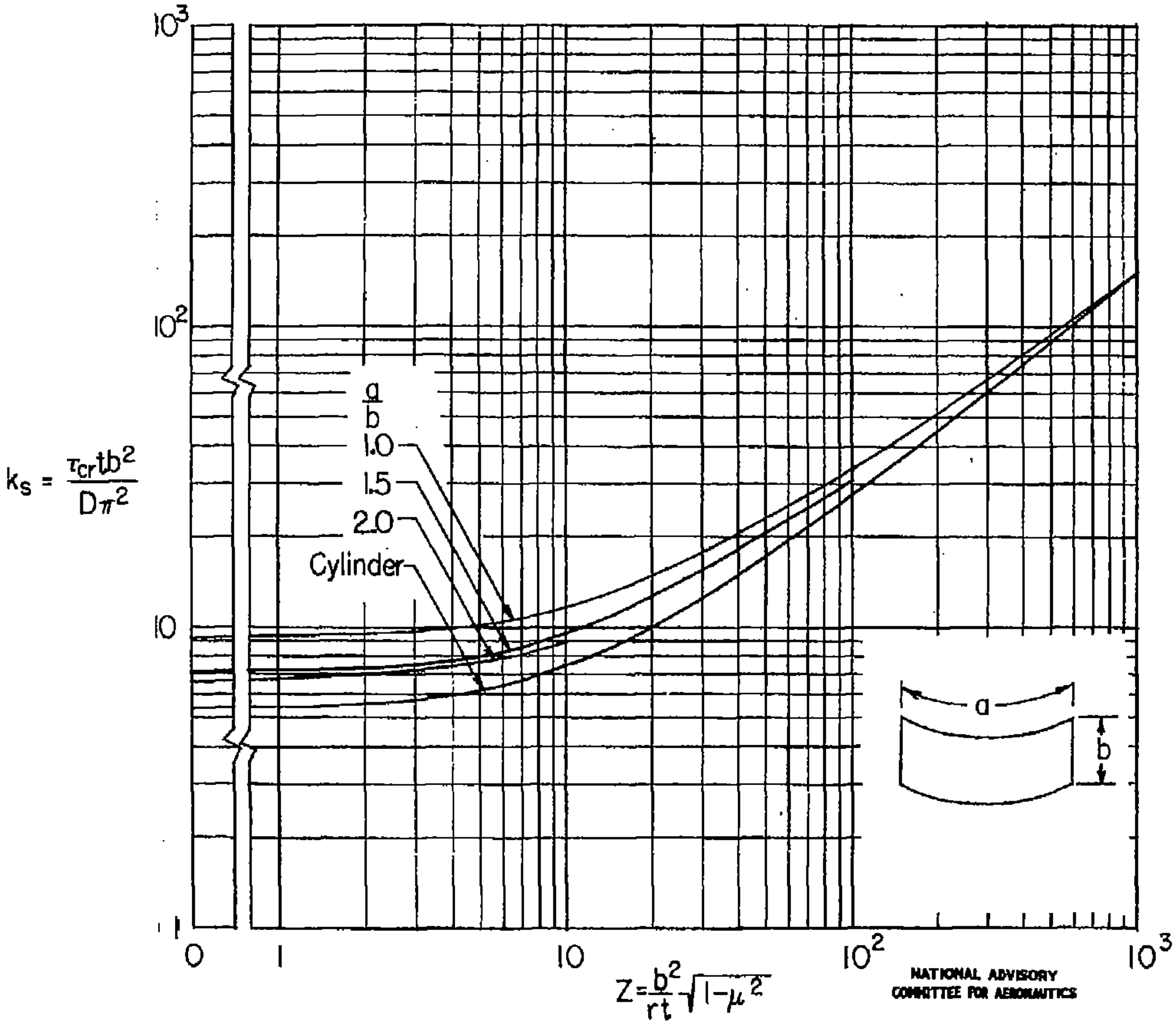


Figure 4.- Critical-shear-stress coefficients for simply supported curved panels having circumferential dimension greater than axial dimension. (Fig. 1 of reference 4.)

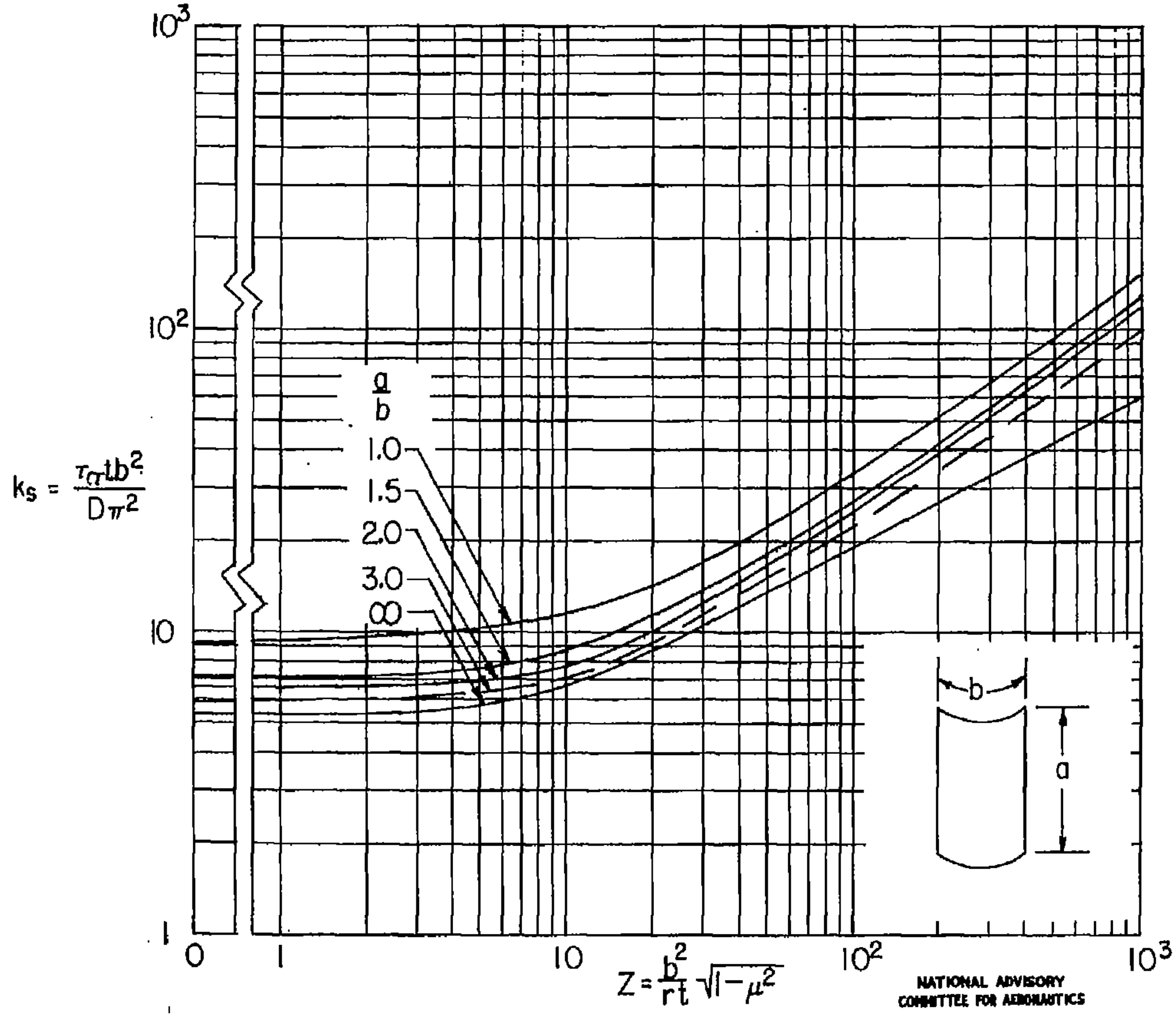
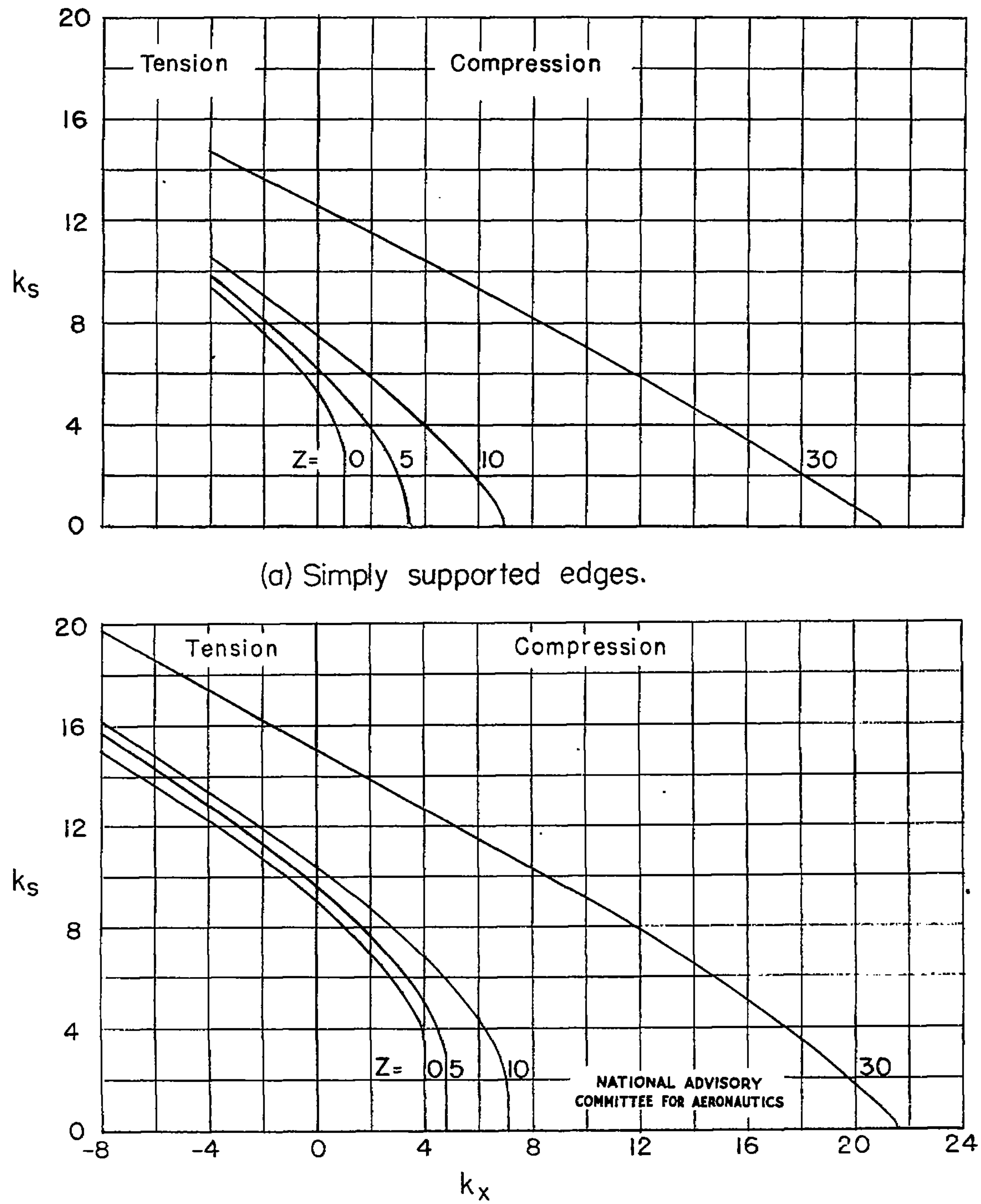


Figure 5.- Critical-shear-stress coefficients of simply supported curved panels having axial dimension greater than circumferential dimension. (Dashed curve estimated.) (Fig. 2 of reference 4.)



(a) Simply supported edges.

(b) Clamped edges.

Figure 6.- Critical combinations of shear-stress and direct-axial-stress coefficients for cylinders. (Fig. 1 of reference 6.)